N/D Method for Complex Partial Waves*

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Conditions on complex partial-wave amplitudes necessary and sufficient for the validity of the Mandelstam representation have been derived for both nonrelativistic potential scattering and relativistic two-particle scattering. These conditions have been used to obtain iterative solutions to the nonrelativistic scattering problem with a given potential (restricted to superposition of Yukawa potentials), and to the relativistic problem in the elastic unitarity approximation with a given discontinuity across the left-hand cut. In the first case, the method used reduces to the well-known determinantal method for physical partial waves, and in the second case, it is a natural generalization of the N/D method to complex values of l. The Regge poles, given by the zeros of the denominator, can be studied in a perturbative expansion in terms of the strength of the potential. Several applications are pointed out.

I. INTRODUCTION

In this paper, we develop a method for the determination of both the nonrelativistic and relativistic partial-wave amplitudes. For the nonrelativistic case, this method has some similarity to the well-known determinantal method,[1] and for the relativistic case, it can be considered as the natural generalization of the N/D method.[2] The approach we are going to use relies heavily on the concept of complex angular momentum, which, since its original introduction,[3] has proved to be a powerful tool in the study of the two-particle scattering matrix. The connection between the poles in the complex angular momentum plane and the high-energy behavior of the scattering amplitude has been made clear in several papers.[4,5] It is also possible, however, to use the notion of complex angular momentum as a practical tool in determining the scattering amplitude from the usual requirements in both the nonrelativistic and relativistic cases. The requirements we have in mind are the Mandelstam representation and the two-particle unitarity relation. In the case of a potential scattering problem with a superposition of Yukawa potentials, these conditions are known to be completely equivalent to the usual Schrödinger equation.[6] As for the relativistic problem, at our present stage of knowledge it seems necessary to assume that the two-particle unitarity condition is exact for all energies and also that the discontinuity across the left-hand cut is a known quantity in order to make the problem tractable. It may still be possible to determine this discontinuity from the crossing relations in a self-consistent fashion[7]; however, we do not concern ourselves about this point in the present paper. Following the approach of an earlier paper,[8] we will obtain conditions on the complex partial waves that are completely equivalent to the Mandelstam representation and the unitarity equation for the original amplitude, for both the relativistic and the nonrelativistic cases. We will then be able to obtain an iterative solution for the partial waves using the N/D method, assuming that the potential in the nonrelativistic case or the discontinuity across the left-hand cut in the relativistic case is given. The poles in the complex angular momentum plane correspond to the zeros of the denominator, and they can be studied by means of a perturbative treatment such as used by Lee and Sawyer in their discussion of the Bethe-Salpeter equation.[9]

II. PROPERTIES OF PARTIAL WAVE AMPLITUDES

In this section, we derive certain conditions for complex partial waves which will ensure the existence of a double-dispersion relation for the total amplitude. In the case of potential scattering, these conditions were first obtained by Bottino *et al.* starting from the Schrödinger equation.[10] In our case, we take the Mandelstam representation as our starting point to simplify matters. We assume that we are dealing with the scattering of a spinless particle from a potential given by

$$rV(r)=g^2\int_a^\infty dk\,\phi(k)\exp(-kr),\quad \text{where}\quad a>0,$$

and we denote the scattering amplitude by $f_p(s_p,t_p)$, where s_p is the square of the energy and t_p is the square of the momentum transfer, and the cosine of the scattering angle is given by $z_p=1-t_p/(2s_p)$. Throughout this paper, the indices p and r will refer to corresponding quantities in potential scattering and relativistic scattering, respectively. Accordingly, f_r, s_r, and t_r will denote the scattering amplitude, center-of-mass energy, and momentum transfer squared, respectively, in the relativistic case, with $z_r=1+2t_r/(s_r-4m^2)$, and for the sake of simplicity, we only consider scattering of identical pseudoscalar particles of mass m. The double-

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dispersion relation for $f_p(s_p,t_p)$ reads:

$$f_p(s_p,t_p)=-\int_a^\infty dt'\frac{g^2\phi(t')}{t'^2+t_p}+(t_p)^{N_p}\int_0^\infty\frac{ds'}{\pi}\int_{a^2}^\infty\frac{dt'}{\pi}\frac{\rho_p(s',t')}{t'^{N_p}(t'+t_p)(s'-s_p)}. \quad (1)$$

Here N_p is chosen sufficiently large to make the integral convergent, and also poles associated with possible bound states and the subtraction polynomial in t_p are suppressed to simplify writing. The relativistic amplitude f_r satisfies the following analogous relation:

$$f_r(s_r,t_r)=s_r^{N_r}t_r^{N_r}\int_{4m^2}^\infty\int_{4m^2}^\infty\frac{ds'}{\pi}\frac{dt'}{\pi}\frac{\rho_r(s',t')}{s'^{N_r}t'^{N_r}(s'-s_r)(t'-t_r)}+s_r^{N_r}u_r^{N_r}\int_{4m^2}^\infty\int_{4m^2}^\infty\frac{ds'}{\pi}\frac{du'}{\pi}\frac{\rho_r(s',u')}{s'^{N_r}t'^{N_r}(s'-s_r)(t'-t_r)}$$
$$+t_r^{N_r}u_r^{N_r}\int_{4m^2}^\infty\int_{4m^2}^\infty\frac{dt'}{\pi}\frac{du'}{\pi}\frac{\rho_r(t',u')}{t'^{N_r}u'^{N_r}(t'-t_r)(u'-u_r)}, \quad (1')$$

where $u_r=4m^2-s_r-t_r$ and the subtraction polynomials in s_r, t_r, and u_r are suppressed as in (1).

In both cases, the partial-wave amplitudes are given by

$$a_l^{(p,r)}(s_{p,r})=\frac{1}{2}\int_{-1}^{1}dz_{p,r}\,P_l(z_{p,r})f_{p,r}(s_{p,r},t_{p,r}), \quad (2)$$

and they can be continued to complex values of l by means of the following defining equations

$$a_p(s_p,l)=\frac{1}{2s_p}\int_{a^2}^\infty\frac{dt}{\pi}Q_l\left(1+\frac{t}{2s_p}\right)A_p(s_p,t), \quad (3)$$

$$a_r(s_r,l)=\frac{4}{s_r-4m^2}\int_{4m^2}^\infty\frac{dt}{\pi}Q_l\left(1+\frac{2t}{s_r-4m^2}\right)A_r(s_r,t), \quad (3')$$

where A_r and A_p are defined to satisfy

$$f_p(s_p,t_p)=\frac{1}{\pi}t_p^{N_p}\int_{a^2}^\infty dt'\frac{A_p(s_p,t')}{t'^{N_p}(t'+t_p)}, \quad (4)$$

and

$$f_r(s_r,t_r)=\frac{1}{\pi}t_r^{N_r}\int_{4m^2}^\infty dt'\frac{A_r(s_r,t')}{t'^{N_r}(t'-t_r)}+\frac{1}{\pi}u_r^{N_r}\int_{4m^2}^\infty du'\frac{A_r(s_r,u')}{u'^{N_r}(u'-u_r)}. \quad (4')$$

In Eqs. (3) and (3′), the real part of l should be taken sufficiently large to make the integrals in question convergent. Equation (3) can easily be shown to be equivalent to Regge's original definition of partial waves in complex l plane.[8]

We finally write down the Watson-Sommerfeld integral[3] which expresses the scattering amplitude in terms of partial waves in a compact form:

$$f_p(s_p,t_p)=\sum_{l=0}^{N_p-1}(2l+1)a_l^{(p)}(s_p)P_l(z_p)-\frac{1}{2i}\int_{N_p-i\infty}^{N_p+i\infty}dl\frac{2l+1}{\sin(\pi l)}a_p(s_p,l)P_l(-z_p), \quad (5)$$

$$f_r(s_r,t_r)=\sum_{l=0}^{N_r-1}(2l+1)a_l^{(r)}(s_r)P_l(z_r)-\frac{1}{4i}\int_{N_r-i\infty}^{N_r+i\infty}dl\frac{2l+1}{\sin(\pi l)}a_r(s_r,l)\times[P_l(z_r)+P_l(-z_r)]. \quad (5')$$

The integrals that appear in (5) and (5′) converge for all complex values of t if the energy is restricted to its physical values ($s_p>0$, $s_r>4m^2$), and they, in general, diverge for complex values of s.

With these preliminaries out of the way, we ask the following question: What are the conditions that must be imposed on $A_{p,r}(s_p,r,l)$ in order that the scattering amplitudes given by (5) and (5′) exist and satisfy the dispersion relations given by (1) and (1′), respectively? Here we definitely ignore the symmetry of ρ_r in (1′) under the interchange of s' and t' and consider only the analyticity and behavior at infinity implied by (1′). (The symmetry mentioned above, which leads to the well-known crossing relations, seems to be very difficult to exploit in an approach which deals with partial waves in a given channel.) With this point in mind, we can now write down the following three conditions on $A_{p,r}$ which are equivalent to (5) and (5′), respectively:

(a) The function $s_p^{(-l)}A_p(s_p,l)$ is analytic in the s_p plane except for cuts on the real axis extending from 0 to $+\infty$ and from $-a^2/4$ to $-\infty$, and in the l plane to the right of a certain line $\mathrm{Re}(l)=N_r$.

(a′) The function $(s_r-4m^2)^{-l}A_r(s_r,l)$ is analytic in the s_r plane except for cuts on the real axis extending from $4m^2$ to $+\infty$ and from 0 to $-\infty$, and in the l plane to the right of a certain line Re $l=N_r$.

(b) As $|l|\to\infty$ in its domain of analyticity,

$$|a_p(s_p,l)|<\frac{C_1}{|s_p|}\left|Q_l\left(1+\frac{a^2}{2s_p}\right)\right|,$$

when $\mathrm{Im}(s_p)$ and $\mathrm{Im}(l)$ have the same sign, and

$$|a_p(s_p,l)|<\frac{C_2}{|s_p|}|e^{\pm i\pi l}|,$$

when $\mathrm{Im}(s_p)$ and $\mathrm{Im}(l)$ have opposite signs. (In the above inequality, the increasing exponential has to be chosen.) In these inequalities, s_p is kept fixed and l is let to vary. The constants C_1 and C_2 may depend on s_p.

(b′) As $|l| \to \infty$ in its domain of analyticity,

$$|a_r(s_r,l)| < \frac{C_1}{|s_r-4m^2|}\left|Q_l\left(1+\frac{8m^2}{s_r-4m^2}\right)\right|,$$

when $\mathrm{Im}(s_r)$ and $\mathrm{Im}(l)$ have the same sign, and,

$$|a_r(s_r,l)| < \frac{C_2}{|s_r-4m^2|}|e^{\pm i\pi l}|,$$

when $\mathrm{Im}(s_r)$ and $\mathrm{Im}(l)$ have opposite signs. Remarks similar to those in (b) also apply here.

(c) For $s_p < -a^2/4$, we have

$$e^{-i\pi l}a_p(s_p+i\epsilon,\, l)-e^{i\pi l}a_p(s_p-i\epsilon,\, l)=E(s_p,l),$$

where $E(s,l)$ is an entire function of l and satisfies:

$$E(s,l)=E(s,\,-l-1).$$

(c′) For $s_r<0$,

$$e^{-i\pi l}a_r(s_r+i\epsilon,\, l)-e^{i\pi l}a_r(s_r-i\epsilon,\, l)=I(s_r,l)+F(s_r,l),$$

where

$$H(s_r,l)=\tan(\pi l)[I(s_r,l)-I(s_r,\,-l-1)]$$

is an entire function of l for fixed s_r, and $F(s_r,l)$ goes to zero as $|l| \to \infty$ uniformly along any direction in its domain of analyticity. Also, for $-4m^2<s_r<0$, $F(s_r,l)$ vanishes and $I(s_r,l)$ becomes an entire function of l, satisfying $I(s_r,l)=I(s_r,\,-l-1)$.

At this point, it must be noted that the bounds (b) and (b′) are certainly not the best possible results. They are, however, sufficient for our purposes.

We now briefly sketch the derivation of these results from (1) and (1′), and we refer the reader to reference 10 for a treatment of the nonrelativistic case starting from Schrödinger's equation. Since (a) and (a′) by now are well known,[10,8,11,12] we restrict ourselves to the second and third conditions. We first assume that s_p is not on the negative real axis and use the fact that the weight function in (3) must be bounded by an expression of the form t^{N_p} uniformly in s_p, and obtain,

$$|a_p(s_p,l)| < \frac{1}{2|s_p|}\int_{a^2}^{\infty}\frac{dt}{\pi}\left|Q_l\left(1+\frac{t}{2s_p}\right)t^{N_p+2}\right|$$
$$\times|A_p(s_p,t)t^{-N_p-2}| < \frac{D}{2|s_p|}\left|Q_l\left(1+\frac{t_0}{2s_p}\right)t_0^{N_p+2}\right|, \quad (6)$$

where t_0 is such that the function Q_l achieves its maximum value at this point, and D is given by the integral over A_p. Now we use the following asymptotic estimates,

$$|Q_l(z)| < \left|\frac{C(z)}{l^{1/2}}[z+(z^2-1)^{1/2}]^l\right|$$
$$\text{as}\quad |l| \to \infty \quad \text{if} \quad z\neq\pm1. \quad (7)$$
$$|Q_l(z)| < |K(l)z^{-l-1}| \quad \text{for} \quad |z| \to \infty.$$

Here the value of the square root is uniquely determined from the condition that $\arg[z+(z^2-1)^{1/2}]$ is never greater than π in absolute value and its sign is the opposite of that of $\mathrm{Im}(z)$. It is then easily seen that when $\mathrm{Im}(s_p)$ and $\mathrm{Im}\,l$ have the same sign, $|Q_l(1+t_0/(2s_p))t_0^{N_p+2}|$ decreases exponentially as $|l| \to \infty$, and the smallest rate of decrease comes from the smallest possible value of t_0, $t_0=a^2$. This gives us the first bound in (b). On the other hand, when $\mathrm{Im}(s_p)$ and $\mathrm{Im}(l)$ have opposite signs, $Q_l(1+t_0/(2s_p))$ blows up exponentially for $|l| \to \infty$, but it is still bounded by $|\exp(\pm i\pi l)|$ for all values of t_0. This gives us the second bound. If s_p is real and negative, the expression $1+t/(2s_p)$ assumes the value -1 for some t, and since $Q_l(z)$ has a logarithmic singularlty at $z=-1$, the above argument fails. One can, however, explicitly separate this logarithmic singularity and still obtain the same results even for the case of negative s_p. The justification of (b′) is completely similar to that of (b) and we do not repeat it here. A possible objection to the reasoning used above is the fact that $A(s,t)$ is not a function but a distribution. We believe, however, that a more careful treatment would yield the same final results.[13]

The conditions (c) and (c′) are easy to derive. The properties of the function Q_l that are needed are the relations

$$Q_l(z\pm i\epsilon)=-e^{\mp i\pi l}Q_l(-z\mp i\epsilon).$$
$$e^{i\pi l}Q_l(z+i\epsilon)-e^{-i\pi l}Q_l(z-i\epsilon) \quad (8)$$
$$=-i\pi P_l(-z). \quad (-1<z<1).$$

Now a direct calculation using (3) and the fact that $A_p(s_p,t)$ has no left-hand cut in the s_p plane immediately gives:

$$e^{-i\pi l}a_p(s_p+i\epsilon,\, l)-e^{i\pi l}a_p(s_p-i\epsilon,\, l)$$
$$=\frac{i}{2s_p}\int_{a^2}^{-4s_p}dt\, P_l\left(-1-\frac{t}{2s_p}\right)A_p(s_p,t).$$
$$(s_p<-a^2/4). \quad (9)$$

From the well-known properties of the function P_l ($P_l(z)=P_{-l-1}(z)$), and from the fact that the range of integration is finite, (c) easily follows. In the relativ-

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istic case, we obtain a similar relation

$$e^{-i\pi l}a_r(s_r+i\epsilon,l)-e^{i\pi l}a_r(s_r-i\epsilon,l)$$
$$=\frac{4}{s_r-4m^2}\int_{4m^2}^{4m^2-s_r}dt\left[Q_l\left(-1-\frac{2t}{s_r-4m^2+i\epsilon}\right)A_r(s_r+i\epsilon,t)-Q_l\left(-1-\frac{2t}{s_r-4m^2-i\epsilon}\right)A_r(s_r-i\epsilon,t)\right]-\frac{4}{s_r-4m^2}$$
$$\times\int_{4m^2-s_r}^{\infty}dtQ_l\left(-1-\frac{2t}{s_r-4m^2}\right)[A_r(s_r+i\epsilon,t)-A_r(s_r-i\epsilon,t)].\quad(s_r<0)\qquad(9')$$

If we identify the first term on the right-hand side with $I(s_r,l)$ and the second term with $F(s_r,l)$, use some well-known properties of Legendre functions and note the fact that for $-4m^2<s_r<0$, $A_r(s_r+i\epsilon,t)-A_r(s_r-i\epsilon,t)$ vanishes, (c′) easily follows.

Our next task is to prove that conditions (a) through (c) with their primed counterparts are actually sufficient for the validity of the Mandelstam representation. The analyticity domain given by (a) plus the fact that $a_p(s_p,l)$ goes to zero as $|l|\to\infty$ for real $s_p>0$ already implies the dispersion relation in momentum transfer given by (4).[3] A similar conclusion also holds in the relativistic case. The proof of a dispersion relation in the energy variable is a little bit more involved. We will give the argument only for the relativistic case, since the nonrelativistic case has already been treated in reference 10, and since it is possible to view the nonrelativistic problem as the limiting case of the relativistic problem in which the left-hand cut in the energy plane vanishes. We now write formula (5′) in a slightly different fashion

$$f_r(s_r,t_r)=\sum_{l=0}^{N_r-1}(2l+1)a_l^{(r)}(s_r)P_l(z_r)+\bar{f}(s_r,t_r)+\bar{f}(s_r,u_r),\qquad(10)$$

where,

$$\bar{f}(s_r,t_r)=-\frac{1}{4i}\int_{N_r-i\infty}^{N_r+i\infty}dl\frac{2l+1}{\sin(\pi l)}a_r(s_r,l)P_l(-z_r).$$

In what follows, we consider only $\bar{f}(s_r,t_r)$, since the other term can be treated in exactly the same fashion by simply interchanging t_r with u_r. At this point, it is convenient to use a slightly different version of the Watson-Sommerfeld integral due to Bottino *et al.*[10]:

$$f(s_r,t_r)=-\frac{1}{4i}\int_{N_r-i\infty}^{N_r+i\infty}dl\frac{2l+1}{\sin(\pi l)}e^{\mp i\pi l}a_r(s_r,l)P_l(z_r).\qquad(11)$$

Formally, (11) can be obtained from the partial wave expansion like the usual Watson-Sommerfeld integral by changing the sign of z_r and compensating for this change with a factor $\exp(\mp i\pi l)$. It remains to investigate the convergence of the integral in question. For this purpose, we take the upper sign in the factor $\exp(\mp i\pi l)$ for $\mathrm{Im}s_r>0$, and the lower sign for $\mathrm{Im}s_r<0$. Using condition (b′), it follows that the integral in (11) converges for all complex s_r if $0<t_r<4m^2$. (See Appendix A for details.) For the same restricted interval in momentum transfer, $\bar{f}(s_r,t_r)$ is then analytic and bounded at infinity by a polynomial in the s_r plane except for possible cuts along the real axis. To investigate these cuts, take $s_r=$ real and form the difference:

$$\bar{f}(s_r+i\epsilon,t_r)-f(s_r-i\epsilon,t_r)$$
$$=-\frac{1}{4i}\int_{N_r-i\infty}^{N_r+i\infty}dl\frac{2l+1}{\sin(\pi l)}P_l(z_r)\times[e^{-i\pi l}a_r(s_r+i\epsilon,l)-e^{i\pi l}a_r(s_r-i\epsilon,l)]$$
$$=-\frac{1}{4i}\int_{N_r-i\infty}^{N_r+i\infty}dl\frac{2l+1}{\sin(\pi l)}P_l(z_r)[I(s_r,l)+F(s_r,l)].\qquad(12)$$

If $-4m^2<s_r<4m^2$, $F(s_r,l)=0$, and for $0<s_r<4m^2$, $I(s_r,l)=0$, [Condition (a′)], hence (12) vanishes. For $-4m^2<s_r<0$, $I(s_r,l)$ is an entire function of l, so that we can shift the line of integration from $\mathrm{Re}l=N_r$ to $\mathrm{Re}l=-1/2$, picking up residues at integer points in between. However, $I(s_r,l)$ vanishes at these points from our original definition, and there is no contribution. Furthermore, on the line $\mathrm{Re}l=-1/2$ the integrand is odd under the substitution $l\to(-l-1)$ since $I(s_r,l)=I(s_r,-l-1)$, and the whole integral therefore vanishes. Hence, the cuts in the s_r plane extend from $s_r=-4m^2$ to $s_r=-\infty$ and from $s_r=4m^2$ to $s_r=\infty$.

To finish the proof, we need yet another version of the Watson-Sommerfeld integral, which we obtain from (12), keeping $s_r<-4m^2$ and $0<t_r<4m^2$, as follows:

$$\bar{f}(s_r+i\epsilon,t_r)-f(s_r-i\epsilon,t_r)$$
$$=-\frac{1}{4i}\int_{N_r-i\infty}^{N_r+i\infty}dl\frac{2l+1}{\sin(\pi l)}P_l(z_r)[I(s_r,l)+F(s_r,l)]$$
$$=-\frac{1}{4i}\int_{N_r-i\infty}^{N_r+i\infty}dl\,\frac{2l+1}{\sin(\pi l)}P_l(z_r)F(s_r,l)-\frac{1}{8i}\int_{-\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}dl\,(2l+1)\frac{\cos(\pi l)}{\sin^2(\pi l)}P_l(z_r)H(s_r,l)$$
$$=-\frac{1}{4i}\int_{N_r-i\infty}^{N_r+i\infty}dl\,\frac{2l+1}{\sin(\pi l)}P_l(z_r)F(s_r,l)-\frac{1}{8\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}dl\,\frac{2l+1}{\sin(\pi l)}[Q_l(z_r)-Q_{-l-1}(z_r)]H(s_r,l)$$
$$=-\frac{1}{4i}\int_{N_r-i\infty}^{N_r+i\infty}dl\,\frac{2l+1}{\sin(\pi l)}P_l(z_r)F(s_r,l)+\frac{1}{2\pi}\sum_{l=0}^{\infty}(2l+1)(-1)^lQ_l(z_r)H(s_r,l).\qquad(13)$$

Here $H(s_r,l)=\tan(\pi l)\{I(s_r,l)-I(s_r,-l-1)\}$ is an entire function in l by (c′). The last step in (13) follows by a residue expansion, and the series in question converges because of the bound on $H(s_r,l)$ at $l=\infty$, given by (b′). Furthermore, since $Q_l(z)$ is a bounded function of z for large z and $\mathrm{Re}\,l>-1/2$, this series continues to converge for complex t_r. The integral involving $f(s_r,l)$ also converges for all complex t_r since $F(s_r,l)$ is a bounded function of l as $|l|\rightarrow\infty$. Since (13) holds for unrestricted values of t_r, the singularities of the left-hand side are given by the singularities of the functions $P_l(z_r)$ and $Q_l(z_r)$, which is a cut on the real axis from $z_r=1$ to $z_r=-\infty$. It can be shown by a simple calculation using the threshold behavior of $a_r(s_r,l)$ that the cut actually starts at $z_r=1+8m^2/(s_r-4m^2)\cdot(s_r<-4m^2)$. It then follows that the jump across the left-hand cut of the function $f_r(s_r,t_r)$ is analytic in the variable t_r except for a cut from $t_r=4m^2$ to $t_r=\infty$. We have mentioned before that (5′) implies that for physical values of s_r, $f(s_r,t_r)$ is analytic in t_r except for cuts from $t_r=4m^2$ to $t_r=\infty$ and from $t_r=-s_r$ to $t_r=-\infty$. It must also be mentioned that as in reference 3, the various integral representations we have, imply that there are only a finite number of subtractions at infinity in the variables of interest, and they have the proper analyticity properties in the remaining variable. Combining the results that follow Eqs. (5′), (11), and (13), we have shown that for $0<t_r<4m^2$, $f_r(s_r,t_r)$ satisfies a dispersion relation in the energy variable, and the spectral functions in this representation themselves satisfy dispersion relations in t_r. From standard results in complex variable theory, this gives us the double-dispersion relation we were looking for. Therefore, we have shown that conditions (a′) through (c′) imply the relation (1′), forgetting about crossing symmetry. A similar proof applies also to the nonrelativistic case, where the left-hand cut of $f_p(s_p,t_p)$ in the s_p plane is clearly seen to vanish as a consequence of (c).

III. POTENTIAL SCATTERING

In this section, we present a method of obtaining the solution of the nonrelativistic scattering problem with superposition of Yukawa potentials, staying within the partial-wave formalism. One can, of course, reduce this problem to the solution of the Schrödinger equation, which can be treated by a number of standard methods. However, it is of some interest to formulate a dispersion-theoretical treatment of potential scattering using only on the mass shell quantities, mainly with the idea of generalizing to the relativistic problem later. In such an approach, the dynamical postulates are the double-dispersion relation and the unitarity condition, and the potential is taken into account as a subtraction in the energy variable. Blankenbecler *et al.*[6] derived a nonlinear integral equation for the double spectral function using these conditions, and they also gave an iteration solution to their equation. Chew and Frautschi later extended this method[7] to the relativistic two-particle scattering in the strip approximation. An unpleasant feature of this approach is the fact that one has to know the number of subtractions at infinity in the momentum-transfer variable right from the start. The alternative procedure we are going to present does not run into this trouble, and it also seems to be particularly well suited to locating the Regge poles since it stays completely within the partial-wave formalism. Our starting points are the conditions (a) through (c) of Sec. II, which are equivalent to the double-dispersion relation, and the unitarity relation for the complex values of angular momentum:

$$a_p(s_p+i\epsilon,l)-a_p(s_p-i\epsilon,l)=2i(s_p)^{1/2}a_p(s_p+i\epsilon,l)a_p(s_p-i\epsilon,l),\quad(14)$$

where $s_p>0$. We now write the partial-wave amplitude in the well-known form:

$$A_p(s_p,l)=\frac{N_p(s_p,l)}{1+D_p(s_p,l)}.\quad(15)$$

The function N_p has only a left-hand cut in the s_p plane and D_p has only a right-hand cut. Since A_p is bounded in l for $s_p>0$, it is clear that the denominator is also bounded in l, and we can normalize it by the condition $D_p\rightarrow 0$ as $l\rightarrow\infty$. Then the numerator itself satisfies (b).

The unitarity relation implies that

$$D_p(s_p,l)=-\frac{1}{\pi}\int_0^\infty\frac{ds'}{s'-s_p}(s')^{1/2}N_p(s_p',l).\quad(16)$$

The integral in (16), and also several integrals we are going to write in what follows, may need one subtraction in s_p to make them convergent. Since this would introduce a trivial modification in our formulas, we are going to ignore it. From (16), one can readily derive the well-known first-order determinantal approximation for the denominator by taking the Born term for the numerator:

$$N_p^{(1)}(s_p,l)=-\frac{g^2}{4s_p}\int_{a^2}^\infty\frac{dt}{t^{1/2}}Q_l\left(1+\frac{t}{2s_p}\right)\phi(t^{1/2}),$$

$$D_p^{(1)}(s_p,l)=\frac{g^2}{2\pi}\int_1^\infty dz\,Q_l(z)\times\int_{a^2}^\infty\frac{dt}{t-2s_p(z-1)}\frac{\phi(t^{1/2})}{[2(z-1)]^{1/2}},\quad(17)$$

where the potential is given by

$$rV(r)=g^2\int_a^\infty dk\,\phi(k)\exp[-kr].$$

To calculate N_p and D_p to higher orders, one has to use the condition on the left-hand cut given by (c). At this point, it simplifies our manipulations considerably to assume that both D_p and N_p can be continued up to the line $\mathrm{Re}\,l=-1/2$. One can then justify this assumption by exhibiting the final answer or, alternatively, one can use the results of reference 3. Condition (c) can now be written in a compact form

$$\Delta\bar{N}_p(s_p,l)=\bar{N}_p(s_p,l)\Delta D_p(s_p,l)-D_p(s_p,l)\Delta\bar{N}_p(s_p,l), \quad (18)$$

where $s_p<-a^2/4$, $\mathrm{Re}\,l=-1/2$, and

$$\bar{N}_p(s_p,l)=\exp(-i\pi l)N_p(s_p+i\epsilon,l)-\exp(i\pi l)N_p(s_p-i\epsilon,l),$$

$$\Delta\bar{N}_p(s_p,l)=1/\pi\tan(\pi l)[\bar{N}_p(s_p,l)-\bar{N}_p(s_p,-l-1)],$$

with a similar expression for ΔD_p.

Equation (18) combined with (16) is already sufficient for the derivation of many interesting results. We, however, first convert (18) to a nonlinear integral equation which is equivalent to the Schrödinger equation. For this purpose, we make the following ansatz about the form of N_p,

$$N_p(s_p,l)=\frac{1}{s_p}\int_{a^2}^{\infty}dtQ_l\left(1+\frac{t}{2s_p}\right)B_p(s_p,t), \quad (19)$$

where

$$B_p(s_p,t)=\int_{a^2/4}^{\infty}ds\frac{\psi_p(s,t)}{s+s_p}+\tilde{\psi}_p(t).$$

It can easily be checked that this ansatz satisfies condition (b) and also has the required domain of analyticity. It is, however, not the most general form N_p can have. It turns out that the fact that N_p can be written as in (19) is closely connected with the analyticity properties of the kinematical factor that appears in the unitarity relation (14). Since we explicitly show that (19) is satisfied, we will not go too deeply into this point here. Substituting (19) into (16), we get

$$D_p(s_p,l)=\int_1^{\infty}dzQ_l(z)C_p(s_p,z),$$

$$C_p(s_p,z)=-\frac{2}{\pi}\int_{a^2}^{\infty}\frac{dt}{t-2s_p(z-1)}\left(\frac{t}{2(z-1)}\right)^{1/2}\times\left[\tilde{\psi}_p(t)+\int_{a^2/4}^{\infty}ds\frac{\psi_p(s,t)}{s+t/2(z-1)}\right]. \quad (20)$$

Furthermore, comparing (19) with the Born approximation given in (17), we get the following simple connection between $\tilde{\psi}_p(t)$ and the function ϕ that appears in the definition of the potential

$$\tilde{\psi}_p(t)=-\frac{g^2}{4t^{1/2}}\phi(t^{1/2}). \quad (21)$$

We can now rewrite (18) using (19) and (20),

$$-2\pi i\int_{-1}^{\infty}dz\,\theta(-2s(z+1)-a^2)P_l(z)\psi_p(-s,-2s(z+1))$$
$$=\int_{-1}^{\infty}dz_1\int_1^{\infty}dz_2\,\theta(-2s(z_1+1)-a^2)C_p(s,z_2)\{[Q_l(z_1+i\epsilon)P_l(z_2)-Q_l(z_2)P_l(z_1)]B_p(s+i\epsilon,-2s(1+z_1))$$
$$-[Q_l(z_1-i\epsilon)P_l(z_2)-Q_l(z_2)P_l(z_1)]B_p(s-i\epsilon,-2s(1+z_1))\}. \quad (s<-a^2/4) \quad (22)$$

To convert (22) into an equation involving only the weight functions, the products of Legendre functions on the right-hand side of (22) should be combined by means of the spherical harmonic addition formula. For $z_1>1$, one can use the following version of the spherical harmonic addition theorem:

$$Q_l(z_1)P_l(z_2)-P_l(z_1)Q_l(z_2)=\int_1^{\infty}dz\,\epsilon(z_2-z_1)\theta(z_1z_2-(z_1^2-1)^{1/2}(z_2^2-1)^{1/2}-z)\frac{P_l(z)}{(z^2+z_1^2+z_2^2-2zz_1z_2-1)^{1/2}}. \quad (23)$$

For a discussion of this formula and other related formulas we are going to use, we refer the reader to Appendix B. In the case $-1<z_1<1$, the situation is a little bit complicated, and the fact that $C_p(s,z_2)$ is analytic in z_2 except for a cut running from $z_2=1$ to $z_2=-\infty$ for $s<-a^2/4$ is needed. In fact, we can write

$$C_p(s,z)=\frac{1}{(z-1)^{1/2}}\int_{-1}^{\infty}dx\frac{\sigma_p(s,x)}{x+z}\quad\text{for}\quad s<-a^2/4,$$

where

$$\sigma_p(s,x)=-\frac{2}{\pi}\theta(-2s(1+x)-a^2)[-s(1+x)]^{1/2}\tilde{\psi}_p(-2s(1+x))-\frac{2}{\pi}\int_{a^2/4}^{\infty}\frac{ds'}{s'+s}\{\theta(-2s(1+x)-a^2)$$
$$\times[-s(1+x)]^{1/2}\psi_p(s',-2s(1+x))-\theta(2s'(1+x)-a^2)(s'(1+x))^{1/2}\psi_p(s',2s'(1+x))\}. \quad (24)$$

For $-1<z_1<1$, we can convert the right-hand side of (22) to the following form by the use of some simple identities[14]:

$$B_p^+[Q_l(z_1+i\epsilon)P_l(z_2)-Q_l(z_2)P_l(z_1)]-B_p^-[Q_l(z_1-i\epsilon)P_l(z_2)-Q_l(z_2)P_l(z_1)]$$

$$=\frac{\pi}{2\sin(\pi l)}[\tfrac{1}{2}(B_p^+-B_p^-)[P_l(-z_2-i\epsilon)+P_l(-z_2+i\epsilon)]P_l(z_1)+\tfrac{1}{2}(B_p^++B_p^-)$$
$$\times[P_l(-z_2-i\epsilon)-P_l(-z_2+i\epsilon)]P_l(z_1)-(B_p^+-B_p^-)P_l(z_2)P_l(-z_1)], \quad (25)$$

with $B_p^+=B_p(s+i\epsilon,\ -2s(1+z_1))$ and similarly for B_p^-. The first terms on the right-hand side of this equation are not suitable for the application of the spherical harmonic addition formula. We get them into a suitable form using (24):

$$\int_1^\infty dz_2\, C_p(s,z_2)\{P_l(-z_2-i\epsilon)+P_l(-z_2+i\epsilon)\}$$
$$=\int_1^\infty dz_2\int_{-1}^\infty dx\,\sigma_p(s,x)\frac{P_l(-z_2-i\epsilon)+P_l(-z_2+i\epsilon)}{(z_2+x)(z_2-1)^{1/2}}=2\pi\int_{-1}^\infty \frac{dx}{(x+1)^{1/2}}\sigma_p(s,x)P_l(x), \quad (26)$$

$$\int_1^\infty dz_2\, C_p(s,z_2)[P_l(-z_2-i\epsilon)-P_l(-z_2+i\epsilon)]=-i\int_{-1}^\infty dz_2\int_{-1}^\infty dx\,\sigma_p(s,x)\frac{P_l(z_2)}{(z_2+1)^{1/2}}\left[\frac{1}{z_2-x+i\epsilon}+\frac{1}{z_2-x-i\epsilon}\right].$$

Substituted into (25), this gives

$$\int_1^\infty dz_2\, C_p(s,z_2)\{B_p^+[Q_l(z_1+i\epsilon)P_l(z_2)-P_l(z_1)Q_l(z_2)]-B_p^-[Q_l(z_1-i\epsilon)P_l(z_2)-P_l(z_1)Q_l(z_2)]\}$$
$$=\frac{\pi}{2\sin(\pi l)}\int_{-1}^\infty dx\left\{\sigma_p(s,x)\left[\frac{\pi}{(x+1)^{1/2}}(B_p^+-B_p^-)P_l(z_1)P_l(x)-\tfrac{1}{2}i(B_p^++B_p^-)\int_{-1}^\infty\frac{dz_2}{(z_2+1)^{1/2}}\right.\right.$$
$$\left.\left.\times\left[\frac{1}{z_2-x+i\epsilon}+\frac{1}{z_2-x-i\epsilon}\right]P_l(z_1)P_l(z_2)\right]-\theta(x-1)(B_p^+-B_p^-)C_p(s,x)P_l(x)P_l(-z_1)\right\}. \quad (-1<z_1<1) \quad (27)$$

The products of various Legendre functions appearing in the above expression can be combined by means of the following addition formula:

$$\frac{P_l(x)P_l(y)}{\sin(\pi l)}=\frac{1}{\pi}\int_{-1}^\infty dz\frac{P_l(z)}{(z^2+x^2+y^2+2xyz-1)^{1/2}}\{\theta(y+x)\theta(z-1)$$
$$+[1+\theta(1-z)]\theta(-y-x)\theta(z+xy-(x^2-1)^{1/2}(y^2-1)^{1/2})\}. \quad (x>-1,\ y>-1,\ \mathrm{Re}\,l=-\tfrac{1}{2}). \quad (28)$$

This addition theorem also immediately leads to the following formulas:

$$\frac{P_l(z_1)}{\sin(\pi l)}\int_{-1}^\infty dz_2\, P_l(z_2)\left[\frac{1}{z_2-x+i\epsilon}+\frac{1}{z_2-x-i\epsilon}\right]$$
$$=\frac{1}{\pi}\int_{-1}^\infty dz\frac{P_l(z)}{V(z,z_1,x)}\left[2\theta(z-1)\ln\left(-\frac{(z-z_1)\epsilon(z-z_1)-x+V(z,z_1,x)}{(z-z_1)\epsilon(z-z_1)-x-V(z,z_1,x)}\right)\right.$$
$$\left.+4\theta(1-z)\theta(z-z_1)\ln\left[\epsilon(x+zz_1+(z_1^2-1)^{1/2}(z^2-1)^{1/2})\frac{U(z,z_1)+zz_1+x+V(z,z_1,x)}{U(z,z_1)+zz_1+x-V(z,z_1,x)}\frac{z-z_1-x+V(z,z_1,x)}{z-z_1-x-V(z,z_1,x)}\right]\right],$$

[14] *Higher Transcendental Functions*, Bateman Manuscript Project (McGraw-Hill Book Company, New York, 1953), Vol. 1, Chap. 3.

$$\frac{P_l(z_1)}{\sin(\pi l)}\int_{-1}^{\infty}\frac{dz_2}{(1+z_2)^{1/2}}P_l(z_2)\left(\frac{1}{z_2-x+i\epsilon}+\frac{1}{z_2-x-i\epsilon}\right)$$

$$=\frac{2}{\pi^2}\int_{-1}^{\infty}dz\int_{1}^{\infty}dy\,\frac{P_l(z)}{(y+x)(y-1)^{1/2}}\left\{\frac{\theta(z-1)}{V(z,z_1,x)}\ln\left(-\frac{(z-z_1)\epsilon(z-z_1)-x+V(z,z_1,x)}{(z-z_1)\epsilon(z-z_1)-x-V(z,z_1,x)}\right)\right.$$
$$+2\frac{\theta(1-z)\theta(z-z_1)}{V(z,z_1,x)}\ln\left[\epsilon(x+zz_1+(z_1^2-1)^{1/2}(z^2-1)^{1/2})\frac{U(z,z_1)+zz_1+x+V(z,z_1,x)}{U(z,z_1)+zz_1+x-V(z,z_1,x)}\right.$$
$$\left.\times\frac{z-z_1-x+V(z,z_1,x)}{z-z_1-x-V(z,z_1,x)}\right]-\frac{\theta(z-1)}{V(z,z_1,-y)}\ln\left(\frac{(z-z_1)\epsilon(z-z_1)+y+V(z,z_1,-y)}{(z-z_1)\epsilon(z-z_1)+y-V(z,z_1,-y)}\right)$$
$$\left.-2\frac{\theta(1-z)\theta(z-z_1)}{V(z,z_1,-y)}\ln\left[\frac{U(z,z_1)+zz_1-y+V(z,z_1,-y)}{U(z,z_1)+zz_1-y-V(z,z_1,-y)}\,\frac{z-z_1+y+V(z,z_1,-y)}{z-z_1+y-V(z,z_1,-y)}\right]\right\},$$

where

$$x>-1,\quad z_1>-1,\quad U(z,z_1)=(z_1^2-1)^{1/2}(z^2-1)^{1/2},\quad V(z,z_1,x)=(z^2+z_1^2+x^2+2xz_1z-1)^{1/2},\tag{29}$$

and the principal branch of the logarithm is to be taken.

Using (28) and (29), one can now transform the right-hand side of (22) into the form

$$\int_{-1}^{\infty}dz\,P_l(z)n(s,z),$$

and since the left-hand side of (22) is already of this form and the equation is to hold for a range of values of l, the corresponding weight functions in the integrals over $P_l(z)$ must be equal. This is the required equation between the weight functions, and to present it in a compact form, we define the following set of functions:

$$K_1(z,x,y)=\int_1^{\infty}dv\,\frac{\epsilon(v-x)\theta(xv-(x^2-1)^{1/2}(v^2-1)^{1/2}-z)}{(v+y)(v^2-1)^{1/2}(z^2+v^2+x^2-2vzx-1)^{1/2}},\quad(x>1).$$

$$K_2(z,x,y)=\int_1^{\infty}dv\,\frac{1}{(v+y)(v-1)^{1/2}(z^2+v^2+x^2-2vzx-1)^{1/2}},\quad(-1<x<1).$$

$$K_3(z,x,y)=\int_1^{\infty}dv\,\frac{1}{(v+y)(v-1)^{1/2}}\left\{\frac{\theta(z-1)}{V(z,x,y)}\ln\left(-\frac{z-x-y+V(z,x,y)}{z-x-y-V(z,x,y)}\right)\right.$$
$$+2\frac{\theta(1-z)\theta(z-x)}{V(z,x,y)}\ln\left[\epsilon(y+zx+(z^2-1)^{1/2}(x^2-1)^{1/2})\frac{U(z,x)+zx+y+V(z,x,y)}{U(z,x)+zx+y-V(z,x,y)}\right.\tag{30}$$
$$\left.\times\frac{z-x-y+V(z,x,y)}{z-x-y-V(z,x,y)}\right]-\frac{\theta(z-1)}{V(z,x,-v)}\ln\left(\frac{z-x+v+V(z,x,-v)}{z-x+v-V(z,x,-v)}\right)$$
$$\left.-2\frac{\theta(1-z)\theta(z-x)}{V(z,x,-v)}\ln\left[\frac{U(z,x)+zx-v+V(z,x,-v)}{U(z,x)+zx-v-V(z,x,-v)}\,\frac{z-x+v+V(z,x,-v)}{z-x+v-V(z,x,-v)}\right]\right\},\quad(-1<x<1,y>-1).$$

$$K_4(z,x,y)=\frac{1}{(y+1)^{1/2}}\,\frac{\theta(x+y)\theta(z-1)+[1+\theta(1-z)]\theta(-x-y)\theta(z+xy-(y^2-1)^{1/2}(x^2-1)^{1/2})}{(z^2+x^2+y^2+2xzy-1)^{1/2}}.\quad(-1<x<1).$$

The first three functions listed can be evaluated in terms of elliptic functions, but we see no advantage in doing so here. Equation (22) can now be written as follows

$$-2i\pi\psi_p(-s,-2s(z+1))$$
$$=\int_{-1}^{\infty}dz_1\,\theta(-2s(1+z_1)-a^2)\int_{-1}^{\infty}dy\sigma_p(s,y)\left\{H_p^-(s,z_1)\left[\frac{\pi}{2}K_4(z,z_1,y)+\theta(z-1)K_1(z,z_1,y)\right.\right.$$
$$\left.\left.-\tfrac{1}{2}\theta(z-1)K_2(z,z_1,y)\right]-\frac{i}{2\pi}K_3(z,z_1,y)H_p^+(s,z_1)\right\},\tag{31}$$

where

$$H_p^-(s,z_1)=B_p(s+i\epsilon,\ -2s(1+z_1))-B_p(s-i\epsilon,\ -2s(1+z_1))=-2\pi i\theta(-2s(1+z_1)-a^2)\psi_p(-s,\ -2s(1+z_1)),$$

$$H_p^+(s,z_1)=B_p(s+i\epsilon,\ -2s(1+z_1))+B_p(s-i\epsilon,\ -2s(1+z_1))$$

$$=2\tilde{\psi}(-2s(1+z_1))+2\int_{a^2/4}^{\infty}\frac{ds'}{s+s'}\left[\psi_p(s',\ -2s(1+z_1))-\left(\frac{4s+a^2}{a^2-4s'}\right)^{1/2}\psi_p(-s,\ -2s(1+z_1))\right].$$

Equation (31) is a nonlinear integral equation in ψ_p alone, since σ_p is expressible in terms of ψ_p through (24), and $\tilde{\psi}_p$ is given in terms of the potential in (21). We can obtain a solution for ψ_p in the form of a power series in g^2 if we start with $\psi_p^{(0)}=0$ as the zeroth term and iterate (31) successively. If this iteration converges, it must yield the correct solution, since it is known that there exists a unique solution to the potential scattering problem. It can be shown that the formal power series obtained in this manner converges by comparing the N/D decomposition with the Jost decomposition, and then using the known results about the expandability of the Jost functions in terms of the coupling constant.[10] Since we are unable to prove this directly, we will not dwell on this point further, and from now on we restrict ourselves to a term-by-term discussion of the solution. To illustrate the method, we exhibit the lowest order contribution to ψ_p, which is proportional to g^4:

$$\psi_p^{(2)}(-s,\ -2s(z+1))$$
$$=-\frac{1}{\pi^2}\int_{-1}^{\infty}dz_1\,\theta(-2s(z_1+1)-a^2)$$
$$\times\int_{-1}^{\infty}dy\,\theta(-2s(y+1)-a^2)[-s(1+y)]^{1/2}$$
$$\times\tilde{\psi}_p(-2s(1+z_1))\tilde{\psi}_p(-2s(1+y))K_3(z,z_1,y).\quad(32)$$

It is of some interest to compare the method presented here with that given in reference 6. Both methods are dispersion theoretical and deal exclusively with quantities on the mass shell. The nonlinear integral equation given in (31) looks more complicated than the corresponding equation of 6, but the procedure given here has the advantage of working directly with partial waves and avoiding the problem of subtractions in the momentum-transfer variable. It may also be noted that the approximation procedure given here satisfies the unitarity relation and has the correct domain of analyticity in momentum transfer at each step of the approximation. Of course, the domain of analyticity in the energy variable at each step is not, in general, the correct domain.

IV. THE RELATIVISTIC PROBLEM

This section is devoted to a treatment of the relativistic problem in analogy with the nonrelativistic one. The statement of the problem is as follows: Determine the relativistic scattering amplitude $f_r(s_r,t_r)$ whose absorptive part $A_r(s_r,t)$ has a known discontinuity across the left-hand energy cut, which we denote by $V(s_r,t)=A_r(s_r+i\epsilon,\ t)-A_r(s_r-i\epsilon,\ t)$ where $s_r<-4m^2$ and $t>4m^2$, and $f_r(s_r,t)$ satisfies the Mandelstam representation and the elastic unitarity condition in the s channel:

$$a_r(s_r+i\epsilon,\ l)-a_r(s_r-i\epsilon,\ l)=2i\left(\frac{s_r-4m^2}{s_r}\right)^{1/2}a_r(s_r+i\epsilon,\ l)a_r(s_r-i\epsilon,\ l).\quad(s_r>4m^2).\quad(33)$$

This problem clearly has no unique solution unless one is also given the arbitrary subtractions in the s and t variables. For the sake of simplicity, we assume that there are no such subtractions; if they are present, it is quite easy to take them into account. We now carry out the N/D decomposition in much the same way as we did before.

$$a_r(s_r,l)=\frac{N_r(s_r,l)}{1+D_r(s_r,l)},$$
$$N_r(s_r,l)=\frac{1}{s_r-4m^2}\int_{4m^2}^{\infty}dt\,B_r(s_r,t)Q_l\left(1+\frac{2t}{s_r-4m^2}\right).\quad(34)$$

$$D_r(s_r,l)=\int_1^{\infty}dz\,C_r(s_r,z)Q_l(z),$$
$$B_r(s_r,t)=\int_0^{\infty}ds\,\frac{\psi_r(s,t)}{s+s_r}.\quad(35)$$

$$C_r(s_r,z)=-\frac{1}{\pi}\int_{4m^2}^{\infty}dt\int_0^{\infty}ds\left(\frac{t}{2m^2(z-1)+t}\right)^{1/2}\times\frac{1}{2t+(z-1)(4m^2-s_r)}\frac{\psi_r(s,t)}{s+4m^2+2t/(z-1)}.\quad(36)$$

For $s_r<0$, $C_r(s_r,z)$ is analytic in the z plane cut from 1 to $-\infty$, and it can be written in the form

$$C_r(s,z)=\int_{-1}^{\infty}dx\,\frac{\sigma_r(s,x)}{x+z},$$

where

$$\sigma_r(s,x)=-\frac{1}{\pi^2}\int_{4m^2}^{\infty}dt\int_0^{\infty}ds'\,\theta(2m^2(x+1)-t)\left(\frac{t}{2m^2(x+1)-t}\right)^{1/2}\frac{\psi_r(s',t)}{[2t-(x+1)(4m^2-s)][s'+4m^2-2t/(x+1)]}$$
$$-\frac{1}{2\pi}\int_0^{\infty}\frac{ds'}{s'+s}\theta(-(x+1)(s-4m^2)-8m^2)\left(\frac{s-4m^2}{s}\right)^{1/2}\psi_r\left(s',-\frac{x+1}{2}(s-4m^2)\right)$$
$$+\frac{1}{2\pi}\int_0^{\infty}\frac{ds'}{s'+s}\theta((x+1)(s'+4m^2)-8m^2)\left(\frac{s'+4m^2}{s'}\right)^{1/2}\psi_r\left(s',\frac{x+1}{2}(s'+4m^2)\right).\quad(37)$$

Equations (34), (35), and (36) clearly satisfy conditions (a′) and (b′) of Sec. II, and it remains to satisfy (c′). Exactly as before, we define the functions $\bar{N}_r(s_r,l)=\exp(-i\pi l)N_r(s_r+i\epsilon,l)-\exp(i\pi l)N_r(s_r-i\epsilon,l)$, $\Delta N_r(s_r,l)=(1/\pi)\tan(\pi l)[N_r(s_r,l)-N_r(s_r,-l-1)]$, and also assume that the functions in question can be continued to the line $\mathrm{Re}l=-1/2$. It is not clear that this assumption is justified in a relativistic theory, but we may hope that our final results are not critically dependent on this assumption. The following equation, valid on the line $\mathrm{Re}l=-1/2$, expresses the whole content of condition (c′):

$$\Delta\bar{N}_r(s_r,l)=\bar{N}_r(s_r,l)\Delta D_r(s_r,l)-\Delta\bar{N}_r(s_r,l)D_r(s_r,l)+[1+D_r(s_r,l)][1+D_r(s_r,-l-1)]L(s_r,l),\quad(38)$$

where $s_r<0$ and

$$L(s_r,l)=-\frac{4}{\pi}\frac{\theta(-s_r-4m^2)}{4m^2-s_r}\int_{4m^2}^{\infty}dt\,P_l\left(-1-\frac{2t}{s_r-4m^2}\right)V(s_r,t).$$

The rest of this section is devoted to converting (43) into an equation between weight functions. First consider the term $\bar{N}_r\Delta D_r-\Delta\bar{N}_rD_r$. This term can be written out in a form similar to (25), and instead of (26), we now have,

$$\int_1^{\infty}dz_2\,C_r(s,z_2)[P_l(-z_2-i\epsilon)-P_l(-z_2+i\epsilon)]=2\pi i\int_{-1}^{\infty}dx\,P_l(x)\sigma_r(s,x),$$
$$\int_1^{\infty}dz_2\,C_r(s,z_2)[P_l(-z_2-i\epsilon)+P_l(-z_2+i\epsilon)]=\int_{-1}^{\infty}dz_2\int_{-1}^{\infty}dx\,\sigma_r(s,x)P_l(z_2)\left[\frac{1}{z_2-x+i\epsilon}+\frac{1}{z_2-x-i\epsilon}\right].\quad(39)$$

We use (28) and the first part of (29) to combine the products of Legendre functions into integrals of a single Legendre function. Next we consider the term $[1+D_r(s,l)][1+D_r(s,-l-1)]$. By a straightforward manipulation, it can be put in the following form:

$$[1+D_r(s,l)][1+D_r(s,-l-1)]$$
$$=1-\frac{\pi}{2\sin(\pi l)}\int_1^{\infty}dz_2\,C_r(s,z_2)[P_l(-z_2-i\epsilon)+P_l(-z_2+i\epsilon)]+\frac{\pi^2}{4\sin^2(\pi l)}\int_1^{\infty}\int_1^{\infty}dz_2dz_2'\,C_r(s,z_2)C_r(s,z_2')$$
$$\times\{-P_l(z_2)P_l(z_2')+\tfrac{1}{2}[P_l(-z_2-i\epsilon)P_l(-z_2'-i\epsilon)+P_l(-z_2+i\epsilon)P_l(-z_2'+i\epsilon)]-2\sin^2(\pi l)P_l(z_2')P_l(z_2)\}.\quad(40)$$

Equation (38) can be transformed into an integral over a single Legendre function exactly as before by the use of (39) and (28). Finally, the product $L(s,l)[1+D_r(s,l)][1+D_r(s,-l-1)]$ can be written in the same fashion using (28) once more. To be able to write the result in a compact form, we define the following functions:

$$M_1(z,x,y)=\int_1^{\infty}dv\,\frac{\epsilon(v-x)\theta(xv-(x^2-1)^{1/2}(v^2-1)^{1/2}-z)}{(v+y)(z^2+v^2+x^2-2vzx-1)^{1/2}}=\frac{1}{V(z,x,y)}\left\{\theta(x-z)\ln\left[\frac{U(z,x)-xz-y+V(z,x,y)}{U(z,x)-xz-y-V(z,x,y)}\right]\right.$$
$$\left.+\theta(x-z)\ln\left[\frac{1+x-z+y+V(z,x,y)}{1+x-z+y-V(z,x,y)}\right]+\ln\left[\frac{U(z,x)+xz+y+V(z,x,y)}{U(z,x)+xz+y-V(z,x,y)}\right]\right\},\quad(x>1).$$

$$M_2(z,x,y)=\frac{2\theta(z-1)}{V(z,x,y)}\ln\left[-\frac{z-x-y+V(z,x,y)}{z-x-y-V(z,x,y)}\right]+\frac{4\theta(1-z)\theta(z-x)}{V(z,x,y)}$$
$$\times\ln\left[\epsilon(y+zx+(z^2-1)^{1/2}(x^2-1)^{1/2})\frac{U(z,x)+zx+y+V(z,x,y)}{U(z,x)+zx+y-V(z,x,y)}\,\frac{z-x-y+V(z,x,y)}{z-x-y-V(z,x,y)}\right],\quad(-1<x<1).$$

$$M_3(z,x,y)=\frac{\theta(x+y)\theta(z-1)+[1+\theta(1-z)]\theta(-x-y)\theta(z+xy-(y^2-1)^{1/2}(x^2-1)^{1/2})}{(z^2+x^2+y^2+2xyz-1)^{1/2}},\quad(-1<x<1).$$

$$M_4(z,x,y)=\int_1^\infty dv\frac{1}{(v+y)(z^2+v^2+x^2-2vzx-1)^{1/2}}=\frac{1}{V(z,x,y)}\ln\left[\frac{1+z-x+y+V(z,x,y)}{1+z-x+y-V(z,x,y)}\right],\quad(-1<x<1).$$

$$M_5(z,x,y,y')=\int_1^\infty dw\int_1^\infty du\int_1^\infty dv\frac{1}{(v+y)(v^2+u^2+x^2+2vux-1)^{1/2}}\frac{1}{(w+y')(u^2+z^2+w^2+2uzw-1)^{1/2}}$$
$$=\int_1^\infty du\frac{1}{V(u,x,y)V(z,u,y')}\ln\left[\frac{1+u-x+y+V(u,x,y)}{1+u-x+y-V(u,x,y)}\right]\ln\left[\frac{1+z-u+y'+V(z,u,y')}{1+z-u+y'-V(z,u,y')}\right].\quad(41)$$

$$M_6(z,x,y,y')=\int_{-1}^\infty dw\int_{-1}^\infty du\int_{-1}^\infty dv\frac{\theta(u-1)\theta(x+v)+[1+\theta(1-u)]\theta(-x-v)\theta(u+xv-(x^2-1)^{1/2}(v^2-1)^{1/2})}{(v^2+u^2+x^2+2vxu-1)^{1/2}}$$
$$\times\frac{\theta(z-1)\theta(u+w)+[1+\theta(1-z)]\theta(-u-w)\theta(z+uw-(u^2-1)^{1/2}(w^2-1)^{1/2})}{(u^2+z^2+w^2+2uzw-1)^{1/2}}$$
$$\times\left[\frac{1}{v-y-i\epsilon}\frac{1}{w-y'-i\epsilon}+\frac{1}{v-y+i\epsilon}\frac{1}{w-y'+i\epsilon}\right]=\int_{-1}^\infty du\{\tfrac{1}{2}M_2(u,x,y)M_2(z,u,y')-2\pi^2[\theta(u-1)+2\theta(1-u)$$
$$\times\theta(u-x)\theta(y+ux+(u^2-1)^{1/2}(x^2-1)^{1/2})][\theta(z-1)+2\theta(1-z)\theta(z-u)\theta(y'+zu+(z^2-1)^{1/2}(u^2-1)^{1/2})]\}.$$

$$M_7(z,x,y,y')=\int_{-1}^\infty dv\frac{\theta(v-1)\theta(x+y)+[1+\theta(1-v)]\theta(-x-y)\theta(v+xy-(x^2-1)^{1/2}(y^2-1)^{1/2})}{(v^2+x^2+y^2+2vxy-1)^{1/2}}$$
$$\times\frac{\theta(z-1)\theta(v+y')+[1+\theta(1-z)]\theta(-v-y')\theta(z+vy'-(v^2-1)^{1/2}(y'^2-1)^{1/2})}{(z^2+v^2+y'^2+2zvy'-1)^{1/2}},$$

where the functions U and V are defined as in (29). Equation (38) now reduces to the following:

$$-2\pi i\psi_r\left(s,\frac{4m^2-s}{2}(z+1)\right)$$
$$=\int_{-1}^\infty dz_1\,\theta((4m^2-s)(z_1+1)-8m^2)\int_{-1}^\infty dy\,\sigma_r(s,y)[\theta(z-1)M_1(z,z_1,y)H_r^-(s,z_1)+\tfrac{1}{4}M_2(z,z_1,y)$$
$$\times H_r^-(s,z_1)+\tfrac{1}{2}\pi iM_3(z,z_1,y)H_r^+(s,z_1)-\tfrac{1}{2}M_4(z,z_1,y)H_r^-(s,z_1)]-\frac{4}{\pi}\theta(-s-4m^2)$$
$$\times\left\{V\left(s,\frac{4m^2-s}{2}(z+1)\right)-\frac{1}{2}\int_{-1}^\infty dz_1\,\theta((4m^2-s)(1+z_1)-8m^2)\int_{-1}^\infty dy\,\sigma_r(s,y)V\left(s,\frac{4m^2-s}{2}(z_1+1)\right)\right.$$
$$\times M_2(z,z_1,y)+\frac{1}{4}\int_{-1}^\infty dz_1\int_{-1}^\infty dy\int_{-1}^\infty dy'\,\theta((4m^2-s)(1+z_1)-8m^2)V\left(s,\frac{4m^2-s}{2}(1+z_1)\right)\sigma_r(s,y)\sigma_r(s,y')$$
$$\left.\times[M_6(z,z_1,y,y')-M_5(z,z_1,y,y')-2\pi^2M_7(z,z_1,y,y')]\right\},\quad(42)$$

where $s<0$ and

$$H_r^-(s,z_1)=B_r\left(s+i\epsilon,\frac{4m^2-s}{2}(1+z_1)\right)-B_r\left(s-i\epsilon,\frac{4m^2-s}{2}(1+z_1)\right)$$
$$=-2\pi i\theta((4m^2-s)(1+z_1)-8m^2)\psi_r\left(-s,\frac{4m^2-s}{2}(1+z_1)\right),$$

$$H_r^+(s,z_1)=B_r\left(s+i\epsilon,\frac{4m^2-s}{2}(1+z_1)\right)+B_r\left(s-i\epsilon,\frac{4m^2-s}{2}(1+z_1)\right)$$

$$=2\int_0^\infty \frac{ds'}{s'+s}\left\{\psi_r\left(s',\frac{4m^2-s}{2}(1+z_1)\right)-\left(-\frac{s}{s'}\right)^{1/2}\psi_r\left(-s,\frac{4m^2-s}{2}(1+z_1)\right)\right\}.$$

Relation (40) can be iterated to yield a series solution in powers of the jump across the left-hand cut. In analogy to the nonrelativistic case, we may hope that this iteration procedure yields convergent results in general.

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APPENDIX A

Here we want to investigate the region of convergence of (11). To this end, we use the asymptotic bound (7) for $P_l(z_r)$ and (b′) for $a_r(s_r,l)$. Let us take $\mathrm{Im}(s_r)\geqq 0$, since the case of negative $\mathrm{Im}(s_r)$ can also be treated similarly. As usual, $[\sin(\pi l)]^{-1}$ is the convergence factor, and the integral converges for all s_r and z_r for which the following inequality is satisfied:

$$-|\arg[z_r+(z_r^2-1)^{1/2}]|+\arg[z_0+(z_0^2-1)^{1/2}]>0, \quad \text{(A1)}$$

where $z_0=1+8m^2/(s_r-4m^2)$. To derive this result, we notice that for $l\to -i\infty$, the second part of (b′) implies that $\exp(-i\pi l)a_r(s_r,l)$ is bounded, and since the factor $P_l(z_r)/\sin(\pi l)$ is always bounded, there is no divergence difficulty for any z_r or s_r. For $l\to +i\infty$, the first part (b′) combined with (7) easily leads to (A1). For $\mathrm{Im}s_r\geqq 0$, $\arg[z_0+(z_0^2-1)^{1/2}]\geqq 0$, and it can easily be verified by simple algebra or a geometrical construction that $|\arg[z_r+(z_r^2-1)^{1/2}]|\leqq|\arg[z_0+(z_0^2-1)^{1/2}]|$ if $z_0-1\leqq z_r-1\leqq 0$. This implies that (11) converges for $0\leqq t_r\leqq 4m^2$.

APPENDIX B

In this section, we derive several versions of the spherical harmonic addition theorem we have been using. The fundamental relations we need are[14]

$$P_l(z)=\frac{\sin(\pi l)}{\pi}\int_1^\infty dx\frac{P_l(x)}{x+z}. \quad \text{(B1)}$$

$$P_l(z_1)P_l(z_2)=\frac{1}{2\pi}\int_0^{2\pi}d\phi \times P_l(z_1z_2-(z_1^2-1)^{1/2}(z_2^2-1)^{1/2}\cos\phi). \quad (z_1>1,\ z_2>1). \quad \text{(B2)}$$

Substituting (B1) into the right-hand side of (B2), and carrying out the integration over ϕ, we easily get,

$$P_l(z_1)P_l(z_2)=\frac{\sin(\pi l)}{\pi}\times\int_1^\infty dx\frac{P_l(x)}{(x^2+z_1^2+z_2^2+2xz_1z_2-1)^{1/2}}. \quad \text{(B3)}$$

This formula, originally valid for $z_1>1$ and $z_2>1$, can be analytically continued to other values of z_1 and z_2 for which the denominator does not vanish. For real z_1 and $z_2>-1$, this imposes the restriction $z_2>-z_1$ and we get the first part of (28). For $-1<z_1<1$ and $z_2<-z_1$, we must first suitably deform the contour of integration in the x plane so as to continue z_2 to values less than $-z_1$ without meeting singularities, and fold back the contour on the real axis, after the continuation is done. This process yields us the extra term in the second formula in (28). Finally, to obtain (23), we use the identity:

$$Q_l(z_1)P_l(z_2)-P_l(z_1)Q_l(z_2) =\frac{1}{2}\frac{\pi}{\sin(\pi l)}[P_l(z_1)P_l(-z_2-i\epsilon) -P_l(z_2)P_l(-z_1-i\epsilon)]. \quad (z_1>1,\ z_2>1). \quad \text{(B4)}$$

Evaluating the right-hand side of (B4) using (B3), we get (23).